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DISPERSION RELATIONS FOR POTENTIAL SCATTERING

by

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ABSTRACT

Dispersion relations for the S-wave scattering amplitude are derived for a class of solvable potentials (superpositions of exponential potentials). Exact solutions are obtained provided that the left-hand singularities (on the complex-energy plane) of the S-wave amplitude are known. For low-energy scattering, approximate solutions, based on the assumption that the contributions from distant singularities are negligible, are treated in terms of the effective-range parameters. As the strength of the potential decreases, the approximate solutions approach the expansion of the exact solutions in powers of both the energy and the strength of the potential.

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Chapter I. INTRODUCTION

One of the most fundamental requirements in a physical theory is the "causality condition" which states that events have a causal relationship only if they are in each other's light cone, that is, only events with time-like separations can be causally related. A violation of this condition, or, more generally, a violation of the postulate that a signal cannot propagate with a velocity greater than that of light, leads to paradoxical consequences.

The first application of the causality condition was made by Kramers[1] and Kronig[2] to the scattering of light by a dispersive medium. They found a relation between the real (dispersive) and imaginary (absorptive) parts of the refractive index. For a general linear system where the connection between the output and a freely variable input is independent of time, Toll[3] has given a rigorous proof of the logical equivalence of the validity of a dispersion relation and the causality condition (no output can occur before the input) which can have a more specific form for a particular physical system. By a dispersion relation we mean a relation

expressing the real part of a generalized scattering amplitude as an integral involving the imaginary part. In relativistic quantum field theory, the causality condition can be satisfied by the commutativity of field operators. If two points x_1 and x_2 have a space-like separation, so that a signal cannot pass between them without exceeding the velocity of light, a measurement of Φ_1 at x_1 cannot interfere with a measurement of Φ_2 at x_2. In such a case, therefore, the two operators would be expected to commute. The causality condition is thus $[\Phi_1(x_1), \Phi_2(x_2)] = 0$ for $(x_1 - x_2)$ space-like. In non-relativistic theory, since there is no maximum velocity, the causality conditon may be interpreted in many ways. For S-wave scattering by a cutoff potential, a dispersion relation for the scattering matrix can be obtained from van Kampen's causality condition[4]. This causality condition implies that the total probability of finding the particle outside the scatterer never exceeds its initial value. Another formulation of the causality condition is Wigner's inequality[5], that the derivative of the scattering phase shift with respect to energy must exceed a certain limit if the interaction of the scattered particle and the scatterer vanishes beyond a certain distance. Saavedra[6], on the other hand,

considered the same problem under the general assumption that the wave functions describing the system form a complete set and recovered not only the dispersion relation for the scattering matrix, but also van Kampen's causality condition and Wigner's inequality. It is well known in quantum mechanics that the Schrödinger equation is to be meaningful only if its solutions form a complete set. Thus, we can say that the causality condition is embodied in the Schrödinger equation. For exponential-tail potentials, Lozano and Moshinsky[7], by using the causality condition that the time-dependent Green's function associated with the system should be bounded for all time, found a dispersion relation for the time-dependent Green's function which is related to the scattering matrix.

In the case of non-relativistic scattering by a potential with exponential range β^{-1}, Khuri[8], using the formal solution of the Schrödinger equation, has shown that a dispersion relation in the energy, s, holds for the scattering amplitude for values of the squared momentum transfer, t, less than or equal to $4\beta^2$. A question arises: can we write a dispersion relation in both s and t for the scattering amplitude which may have a larger region of analyticity in t than that

mentioned by Khuri[8]? Blankenbecler et al.[9], by using Lehmann's method[10] for positive s, have shown that the scattering amplitude, aside from the first Born approximation, is in fact analytic in t inside an ellipse which intersects the real axis at $t = -4\beta^2$ and $t = 4\beta^2 + 4s$. Mandelstam[11] has made the conjecture that the scattering amplitude is an analytic function of both energy and momentum transfer variables with singularities only on the real axes of these two variables, so that one can write down a "double dispersion relation" for the scattering amplitude, the so-called Mandelstam representation. Since then various attempts have been made to prove the existence of such a representation for different scattering processes. Blankenbecler et al.[9], Klein[12], and Regge[13] have shown that the scattering amplitude for non-relativistic scattering by superpositions of Yukawa potentials satisfies a simple Mandelstam representation. We might mention here that a cutoff potential will not satisfy the Mandelstam representation, which isolates the first Born approximation from the scattering amplitude, because the first Born approximation for a cutoff potential has an essential singularity at infinity in the momentum transfer plane. However, Nussenzveig[14] has obtained another kind of double dispersion relation which involves some exponential

factors required to remove that essential singularity at infinity.

We have been talking about the scattering matrix and the scattering amplitude. These two quantities are very closely related to each other through partial-wave phase shifts as can be readily seen from partial-wave analysis for the scattering process[15]. The phase shift of a particular partial wave is the difference in phase between the asymptotic forms of the actual radial function, the solution of the Schrödinger equation with the potential for the particular partial wave, and the free-particle radial function. Hence the effect of the potential is to shift the phase of each outgoing partial wave. At low energies only S-wave scattering is important and the cotangent of the S-wave phase shift can be described by the effective-range parameters[16]. We can attach more physical significance to the scattering amplitude, as it is well known that the integration over all angles for the square modulus of the scattering amplitude yields the total cross section which, in turn, gives a measure of the total probability that the particle of a given energy is scattered in some direction by the potential. In practice we know little about the

potential, but we can experimentally measure the total cross section. From these measurements it is possible to deduce some information about the interaction potential. For forward scattering the imaginary part of the scattering amplitude is related to the total cross section (the optical theorem). In general, however, the imaginary part is given by the unitarity condition of the scattering amplitude. The unitarity condition and the optical theorem are both statements of the conservation of particle flux in the scattering process. As we shall see later, unitarity plays an important rôle in calculating the scattering amplitude.

In this work we shall apply the dispersion-relation technique to a class of solvable potentials which can be expressed as a finite or an infinite sum of exponential potentials (a special case of superpositions of Yukawa potentials)

$$V(r) = \beta^2 \sum_{n=1}^{\infty} a_n \, e^{-n\beta r}, \qquad \text{(I-1)}$$

where $(n\beta)^{-1}$ is the range of the n^{th} part of the potential, which is related to the mass of the "exchanged" particle in analogy with the meson theory of nuclear

forces. Two nucleons interact with each other through virtual exchange of pions; the one-pion exchange processes make a contribution of characteristic range β^{-1} if $\mu = \beta\hbar/c = \beta(\hbar = c = 1)$ is the pion mass. Similarly, n-pion (or one particle of mass $n\mu$) exchange processes will have a characteristic range of $(n\beta)^{-1}$ [17,18]. We assume that the coefficients a_n of (I-1) satisfy the following conditions:

(a) $\sum_{n=1}^{\infty} a_n$ converges absolutely and to a negative number; and

(b) the partial sum of a_n, $\sum_{n=1}^{N} a_n$, is bounded and negative for all values of N.

In fact the boundedness condition is implied by (a) since a convergent series must be bounded. Physically the first statement means that the potential is everywhere finite and attractive; while the second statement implies that the sign (negative) of the potential produced by the exchange of all particles with masses less than $N\beta$ is the same as the sign of the potential produced by the lightest particle with mass β.

We shall consider only S-wave scattering by this class of potentials and exclude cases for which

bound states are present. Behaviour of the S-wave amplitude at low energies is also treated in terms of effective-range parameters.

Chapter II. ANALYTIC PROPERTIES OF THE S-WAVE SCATTERING AMPLITUDE

In order to study the analytic properties of the S-wave scattering amplitude, the Jost function[19] of the Schrödinger equation will be obtained. From the analytic properties of the Jost function, we can then read off the analytic properties of the S-wave amplitude.

II.1 Jost Function and the Scattering Amplitude

The S-wave Schrödinger equation for two particles of equal mass, M, in the centre-of-mass coordinates is

$$\frac{d^2u}{dr^2} + su = V(r)u \,, \qquad \text{(II-1)}$$

where $u = r\Psi$, and $s = k^2 = ME/\hbar^2$ the energy of the particle $(M/\hbar^2 = 1)$. Following Jost[19] let us introduce two independent solutions of (II-1), $g(\sqrt{s},r)$ and $g(-\sqrt{s},r)$, with the properties

$$\lim_{r\to\infty} g(\pm\sqrt{s},r)\exp(\pm i\sqrt{s},r) = 1,$$

$$g^*(-\sqrt{s}^*,r) = g(\sqrt{s},r) \,, \qquad \text{(II-2)}$$

where $g(\pm\sqrt{s},r)$ are the so-called Jost functions. Then an admissible solution of (II-1) obeying the proper boundary condition ($u(0) = 0$ since $u = r\Psi$) is given by

$$u(r) = [g(\sqrt{s},o)\, g(-\sqrt{s},r) - g(-\sqrt{s},0)\, g(\sqrt{s},r)]/2i\,|g(\sqrt{s},o)|\,. \tag{II-3}$$

Asymptotically,

$$u(r) \sim \frac{[g(\sqrt{s},o)\exp(i\sqrt{s}r) - g(-\sqrt{s},o)\exp(-i\sqrt{s}r)]}{2i\;|g(\sqrt{s},o)|} \sim \sin(\sqrt{s}r+\eta),$$

where the phase shift, η , is determined by

$$\exp(2i\eta) = g(\sqrt{s},o)/g(-\sqrt{s},o) = S$$

(the $\ell = o$ component of the scattering matrix); and the S-wave scattering amplitude is

$$\begin{aligned} A(s) &= \exp(i\eta)\,\sin\eta/\sqrt{s} = (S-1)/2i\sqrt{s} \\ &= [g(\sqrt{s},o)/g(-\sqrt{s},o)-1]/2i\sqrt{s}\,. \end{aligned} \tag{II-4}$$

II.2 Analytic Properties of the Jost Function

From (II-1) and (II-2), $g(\sqrt{s},r)$ is given equivalently by the solution of the integral equation,

$$g(\sqrt{s},r) = e^{-i\sqrt{s}r} + \frac{1}{\sqrt{s}} \int_r^\infty \sin\sqrt{s}(r'-r)V(r')g(\sqrt{s},r')dr', \tag{II-5}$$

which, in general, can be solved by iteration. For the potential (I-1) we obtain by iterating (II-5)

$$g(\sqrt{s},r) = \sum_{m=o}^{\infty} g^{(m)}(\sqrt{s},r) , \tag{II-6}$$

where

$$g^{(o)}(\sqrt{s},r) = e^{-i\sqrt{s}r} ,$$

$$g^{(m)}(\sqrt{s},r) = e^{-i\sqrt{s}r} \beta^m \sum_{j_1,j_2\ldots} \cdots\cdots \sum_{j_m=1}^{\infty}$$

$$\times \frac{a_{j_1}\ldots a_{j_m} \exp[-(j_1 + \ldots j_m)\beta r]}{j_1(j_1+j_2)\ldots(j_1+\ldots j_m)[j_1\beta+2i\sqrt{s}][(j_1+j_2)\beta+2i\sqrt{s}]\ldots[(j_1+\ldots j_m)\beta+2i\sqrt{s}]} \tag{II-7}$$

We shall show that both $g^{(m)}(\sqrt{s},r)$ and $g(\sqrt{s},r)$ converge absolutely. The series $g^{(m)}(\sqrt{s},r)$ is essentially the product of m absolutely convergent series of the form

$$\sum_{j=1}^{\infty} \frac{a_j \exp(-j\beta r)}{j(j\beta + 2i\sqrt{s})} \quad ,$$

hence the product series $g^{(m)}(\sqrt{s},r)$ is also absolutely convergent. The ratio test will be used to prove the convergence of $g(\sqrt{s},r)$:

$$\left| \frac{g^{(m+1)}(\sqrt{s},r)}{g^{(m)}(\sqrt{s},r)} \right| =$$

$$\left| \frac{e^{-i\sqrt{s}r} \quad \beta^{m+1} \sum\limits_{j_1} \cdots \sum\limits_{j_m} \sum\limits_{j_{m+1}} \dfrac{a_{j_1} \cdots a_{j_m} a_{j_{m+1}}}{j_1 \cdots (j_1 + \cdots j_m)(j_1 + \cdots j_m + j_{m+1})}}{e^{-i\sqrt{s}r} \quad \beta^m \sum\limits_{j_1} \cdots \sum\limits_{j_m} \dfrac{a_{j_1} \cdots a_{j_m}}{j_1 \cdots (j_1 + \cdots j_m)}} \right.$$

$$\left. \times \frac{\dfrac{\exp[-(j_1 + \cdots j_m + j_{m+1})\beta r]}{[j_1\beta + 2i\sqrt{s}] \cdots [(j_1 + \cdots j_m)\beta + 2i\sqrt{s}][(j_1 + \cdots j_m + j_{m+1})\beta + 2i\sqrt{s}]}}{\dfrac{\exp[-(j_1 + \cdots j_m)\beta r]}{[j_1\beta + 2i\sqrt{s}] \cdots [(j_1 + \cdots j_m)\beta + 2i\sqrt{s}]}} \right|$$

$$< \beta \left| \frac{g^{(m)}(\sqrt{s},r) \sum\limits_{j_{m+1}} \dfrac{a_{j_{m+1}} \exp(-j_{m+1}\beta r)}{(m+j_{m+1})[(m+j_{m+1})\beta + 2i\sqrt{s}]}}{g^{(m)}(\sqrt{s},r)} \right|$$

$$= \beta \left| \sum_j \frac{a_j \exp(-j\beta r)}{(m+j)[(m+j)\beta + 2i\sqrt{s}]} \right|$$

since m is the least possible sum of j_1 through j_m. The last series, $\Sigma \dfrac{a_j \exp(-j\beta r)}{(m+j)[(m+j)\beta+2i\sqrt{s}]}$, converges absolutely for finite m, and in the limit as $m \to \infty$,

$$\left| \frac{g^{(m+1)}(\sqrt{s},r)}{g^{(m)}(\sqrt{s},r)} \right| \to \beta \left| \sum_j \frac{a_j \exp(-j\beta r)}{(m+j)[(m+j)\beta+2i\sqrt{s}]} \right| \to 0 ,$$

therefore, $g(\sqrt{s},r)$ converges absolutely.

The analytic properties of the Jost function can be readily seen from (II-6) and (II-7). For $g(\sqrt{s},o)$ there are poles at the points where $n\beta + 2i\sqrt{s} = 0$, while $g(-\sqrt{s},o)$ has poles at $n\beta - 2i\sqrt{s} = 0$, $n = 1,2,3,\ldots$. Aside from these poles, there are no other

singularities in the entire k plane ($k = \sqrt{s}$). The zeros of $g(-\sqrt{s},o)$, in the upper half k plane, correspond to the bound states[19]. As $s \to \infty$, both $g(\sqrt{s},o)$ and $g(-\sqrt{s},o)$ approach 1; and at $s = o$, $g(\sqrt{s},o) = g(-\sqrt{s},o)$.

II.3 Analytic Properties of the Scattering Amplitude

We shall study the analytic properties of the S-wave amplitude in terms of the energy variable s instead of the wave number $k(=\sqrt{s})$. Let us define a complex variable $z = s + iy$ (the complex energy). In the complex z plane $A(z)$ is an analytic function of z except on the real axis. For positive real values of z, i.e. $s \geqslant o$, $A(s)$ will have a cut from $s = o$ to $s = \infty$ because it depends on $\sqrt{s}$. The discontinuity across the cut is given by the unitarity condition, from (II-4),

$$\text{Im } A(s) = \sin^2\eta/\sqrt{s} = \sqrt{s}\,|A(s)|^2, \quad s \geqslant o. \tag{II-8}$$

To find the singularities for $s < o$, we shall replace the complex z plane by a Riemann surface

of two sheets, linked by a branch point at the origin, each sheet covering the z plane once with a cut along the positive real axis. The first sheet corresponds to $\mathrm{Im}\sqrt{s}$ positive, i.e. the upper half k plane, the physical sheet. The second corresponds to $\mathrm{Im}\sqrt{s}$ negative, the lower half k plane. Our discussion will be limited to the physical sheet, but bound states will be excluded. From the analytic properties of the Jost function, the singularities of $A(s)$ are poles from $g(\sqrt{s},o)$ on the first sheet at the points where $n\beta + 2i\sqrt{s} = 0$, i.e. at

$$s = s_n = -n^2\beta^2/4 \;, \quad n=1, 2, \ldots \qquad \text{(II-9)}$$

It is important to note that all of these poles may not exist, i.e. the residue of $A(s)$ at one or at an infinite number of poles may be equal to zero (removable singularities). Thus, the singularities on the negative s axis start at $s = -\beta^2/4$ and consist of a finite or an infinite number of isolated poles. As $|z| \to \infty$, $A(z) \to 0$. We conclude that aside from the singularities on the real axis, $A(z)$ is analytic in the complex z plane.

Chapter III. INTEGRAL EQUATION FOR THE S-WAVE SCATTERING AMPLITUDE

A dispersion relation for the S-wave scattering amplitude, in the form of a nonlinear integral equation, will be derived from the analytic properties of the S-wave amplitude and from the Mandelstam representation for the total-wave amplitude[9]. The exact solution will be obtained provided that the left-hand singularities are known. We shall also obtain an approximate solution for low-energy scattering in terms of the effective-range parameters.

III.1 Methods of Derivation

From the analytic properties of the S-wave scattering amplitude we shall derive a dispersion relation for $A(s)$ by evaluating the following integral[12],

$$A(z) - f_B(z) = \frac{1}{2\pi i} \oint \frac{A(z') - f_B(z')}{z' - z} \, dz' , \tag{III-1}$$

taken around the contour of Fig. 1 which excludes both the right-hand cut and the poles on the negative s axis.

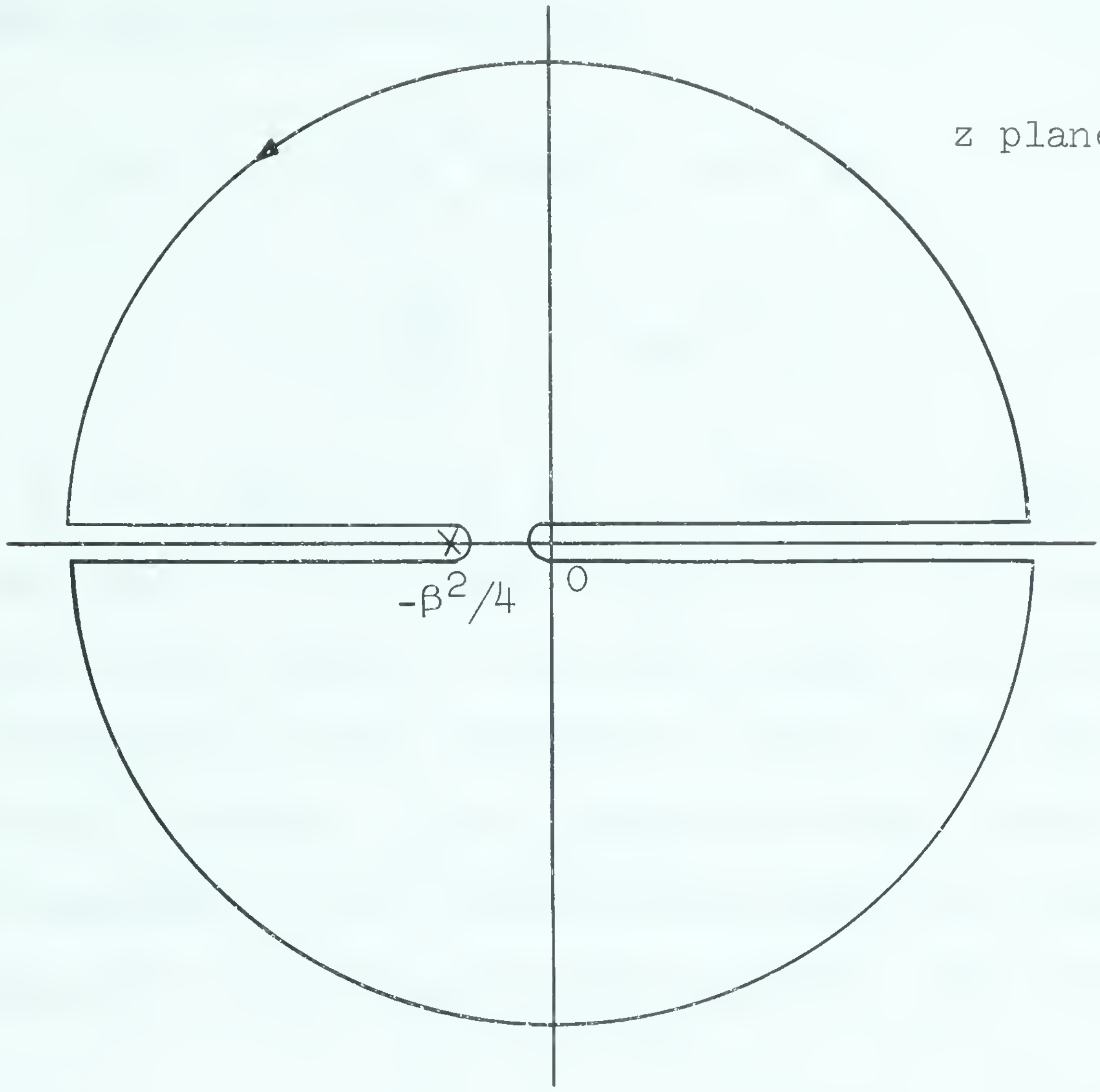

Fig. 1   Contour for the Integration of Equation (III-1)

The variable $z = s + iy$ is the complex energy (Section II.3). For sufficiently large z, $A(z)$ is dominated by its first Born approximation:

$$A(z) \sim \frac{1}{2i\sqrt{z}} \left[\frac{1+\beta\Sigma \dfrac{a_j}{j(j\beta+2i\sqrt{z})}}{1+\beta\Sigma \dfrac{a_j}{j(j\beta-2i\sqrt{z})}} - 1 \right] \approx -\frac{\beta}{2} \sum_{n=1}^{\infty} \frac{a_n}{n(z+n^2\beta^2/4)} .$$

The first Born approximation is

$$f_B(z) = -\frac{1}{z} \int_0^\infty \sin^2 \sqrt{z}\, r\, V(r)\, dr$$

$$= \Sigma \frac{R_n}{z-s_n} = \Sigma f_{Bn}(z) \quad , \qquad \text{(III-2)}$$

where $R_n = -\beta a_n/2n$ and $s_n = -n^2\beta^2/4$. Hence, in the limit as $z \to \infty$, $A(z) - f_B(z) \to 0$. We can then say that as the radius of the circle tends to infinity, the contribution to the integral in (III-1) from the circle will be zero. In the absence of bound states, as the remainder of the contour is deformed into integrals enclosing the cut and encircling the poles, we obtain

$$A(s) = f_B(s) + \int_0^\infty \frac{ds'}{\pi} \frac{\operatorname{Im} A(s')}{s' - s - i\epsilon}$$

$$+ \int_{-\infty}^{-\beta^2} \frac{ds'}{\pi} \frac{\operatorname{Im}[A(s') - f_B(s')]}{s' - s}$$

$$+ \int_{-\beta^2}^{-\beta^2/4} \frac{ds'}{\pi} \frac{\operatorname{Im}[A(s') - f_B(s')]}{s' - s} \quad .$$

(III-3)

The last integral in (III-3) vanishes because in the interval, $-\beta^2 < s \leqslant -\beta^2/4$,

$$\mathrm{Im}\, A(s) = -\pi\delta(s+\beta^2/4)\,[\text{residue of } A(s) \text{ at } s=-\beta^2/4]$$

$$= -\pi\delta(s+\beta^2/4)\,[-\beta a_1/2] = \mathrm{Im}\, f_B(s)\;. \qquad \text{(III-4)}$$

Hence,

$$A(s) = f_B(s) + \int_0^\infty \frac{ds'}{\pi}\frac{\mathrm{Im}A(s')}{s'-s-i\epsilon}$$

$$+ \int_{-\infty}^{-\beta^2} \frac{ds'}{\pi}\,\frac{\mathrm{Im}[A(s') - f_B(s')]}{s' - s} \qquad \text{(III-5)}$$

which is the Mandelstam representation for the S-wave scattering amplitude[9,12].

Another way of obtaining (III-5) is to assume the Mandelstam double dispersion relation for the scattering amplitude of the total wave from which we can then project out the S-wave amplitude[9]. In this

derivation we shall use the following notation:

$A(s,t)$ — total scattering amplitude,

$f_B(t)$ — first Born approximation for the total wave,

$A_\ell(s)$ — scattering amplitude for the ℓ^{th} partial wave,

$f_{\ell B}(s)$ — first Born approximation for the ℓ^{th} partial wave.

If the potential is defined as

$$rV(r) = \int_{\beta^2}^{\infty} \sigma(\mu^2) \exp(-\mu r)\, d(\mu^2) \quad , \tag{III-6}$$

for our potential (I-1), $\sigma(\mu^2) = 2\beta^3 \sum_{n=1}^{\infty} n a_n \delta'(\mu^2 - n^2\beta^2)$.

When there are no bound states we can write the double dispersion relation as[9]

$$A(s,t) = -\int_{\beta^2}^{\infty} dt' \frac{\sigma(t')}{t'+t} + \int_0^\infty \frac{ds'}{\pi} \int_{4\beta^2}^{\infty} \frac{dt'}{\pi} \frac{\rho(s',t')}{(s'-s-i\epsilon)(t'+t)} \quad , \tag{III-7}$$

where $t = 2s(1 - \cos\theta)$ is the square of the momentum transfer and $\rho(s',t')$ is called the double spectral

function. The first term in (III-7) is just the first Born approximation for the total wave,

$$f_B(t) = -\int_{\beta^2}^{\infty} dt' \frac{\sigma(t')}{t'+t} = -2\beta^3 \sum_{n=1}^{\infty} \frac{na_n}{(t+n^2\beta^2)^2} . \tag{III-8}$$

The total amplitude, $A(s,t)$, can be represented by the series

$$A(s,t) = \sum_{\ell=o}^{\infty} (2\ell+1)\, A_\ell(s)\, P_\ell(1 - \frac{t}{2s}). \tag{III-9}$$

We shall apply the following projection operator, $\hat{L}$, to (III-7) to separate the partial-wave amplitude from $A(s,t)$,

$$\hat{L}\, A(s,t) = \frac{1}{2} \int_{-1}^{+1} d(\cos\theta)\, P_\ell(\cos\theta)\, A(s,t) = A_\ell(s). \tag{III-10}$$

In doing so, we encounter integrals of the form

$$\frac{1}{2} \int_{-1}^{+1} d(\cos\theta) \frac{P_\ell(\cos\theta)}{t'+2s(1-\cos\theta)} = \frac{1}{2s} Q_\ell(1 + \frac{t'}{2s}) \tag{III-11}$$

which has a cut, present in the Q_ℓ's (Legendre function of the second kind), along the negative s axis with a discontinuity evaluated from (III-11),

$$\frac{1}{2i}\left[Q_\ell\left(1+\frac{t'}{2(s+i\epsilon)}\right) - Q_\ell\left(1+\frac{t'}{2(s-i\epsilon)}\right)\right] =$$

$$\frac{\pi}{2} P_\ell\left(1+\frac{t'}{2s}\right)\theta\left(-s-\frac{t'}{4}\right), \qquad \text{(III-12)}$$

where $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. For S-wave we obtain, by applying the projection operator to (III-7), the expression

$$A_o(s) = -\int_{\beta^2}^{\infty} dt'\sigma(t')\frac{1}{2s}Q_o\left(1+\frac{t'}{2s}\right)$$

$$+\int_o^\infty \frac{ds'}{\pi}\int_{4\beta^2}^\infty \frac{dt'}{\pi}\frac{\rho(s',t')}{s'-s-i\epsilon}\frac{1}{2s'}Q_o\left(1+\frac{t'}{2s'}\right)$$

$$+\int_o^\infty \frac{ds'}{\pi}\int_{4\beta^2}^\infty \frac{dt'}{\pi}\frac{\rho(s',t')}{s'-s-i\epsilon}\left[\frac{1}{2s}Q_o\left(1+\frac{t'}{2s}\right) - \frac{1}{2s'}Q_o\left(1+\frac{t'}{2s'}\right)\right]$$

(III-13)

where we have added and subtracted a term (the second term). The first term is the first Born approximation for the S-wave, $f_{oB}(s) = -\frac{\beta}{2}\Sigma\frac{a_n}{n(s+n^2\beta^2/4)}$, with a cut

resulting from the $Q_o(1+\frac{t'}{2s})$ extending from $s=-\infty$ to $s=-\beta^2/4$ (degenerated into poles in our case). The second term has a cut along the positive s axis from its denominator; while the third term has no discontinuity across the positive s axis by construction, it has only a left-hand cut associated with $Q_o(1+\frac{t'}{2s})$ from $s=-\infty$ to $s=-\beta^2$ (again degenerated into poles for our potential). The discontinuity across the positive s axis comes only from the second term which can be expressed as

$$\begin{aligned} \mathrm{Im}A_o(s) &= [A_o(s+i\epsilon) - A_o(s-i\epsilon)]/2i \\ &= \int_{4\beta^2}^{\infty} \frac{dt'}{\pi}\, \rho(s,t')\, \frac{1}{2s}\, Q_o(1+\frac{t'}{2s}) \; . \end{aligned} \qquad \text{(III-14)}$$

We note also that, for $-\infty < s \leqslant -\beta^2/4$, using (III-12),

$$\begin{aligned} \mathrm{Im}[A_o(s) - f_{oB}(s)] &= (\text{discontinuity of the third term across the negative s axis})/2i \\ &= \int_0^{\infty} \frac{ds'}{\pi} \int_{4\beta^2}^{\infty} \frac{dt'}{\pi}\, \frac{\rho(s',t')}{s'-s}\, \frac{\pi}{4s}\, P_o(1+\frac{t'}{2s})\, \theta(-s-\frac{t'}{4}) \; . \end{aligned} \qquad \text{(III-15)}$$

Hence we can rewrite (III-13), using (III-14) and (III-15),

$$A_o(s) = f_{oB}(s) + \int_0^\infty \frac{ds'}{\pi} \frac{\text{Im}A_o(s')}{s'-s-i\epsilon}$$

$$+ \int_{-\infty}^{-\beta^2} \frac{ds'}{\pi} \frac{\text{Im}[A_o(s') - f_{oB}(s')]}{s' - s} \tag{III-16}$$

which is what we have obtained before using the analytic properties of the S-wave amplitude. We can show that the last term in (III-16) is identical to the last term in (III-13) by substituting (III-15) and carrying out the integration from $s' = -\infty$ to $s' = -\beta^2$ using (III-11). From (III-16) and (III-15) we have the expression,

$$I = \int_{-\infty}^{-\beta^2} \frac{ds'}{\pi} \frac{\text{Im}[A_o(s') - f_{oB}(s')]}{s' - s}$$

$$= \int_{-\infty}^{-\beta^2} \frac{ds'}{\pi} \frac{1}{s'-s} \int_0^\infty \frac{ds''}{\pi} \int_{4\beta^2}^\infty \frac{dt'}{\pi} \frac{\rho(s'',t')}{s''-s'} \frac{\pi}{4s'}$$

$$\cdot \; P_o(1 + \frac{t'}{2s'}) \; \theta(-s' - \frac{t'}{4}) \quad .$$

The integration over s' involves

$$\frac{1}{\pi}\int_{-\infty}^{-\beta^2} \frac{ds'}{(s'-s)(s''-s')}\,\frac{\pi}{4s'}\, P_o(1+\frac{t'}{2s'})\ \theta(-s'-\frac{t'}{4}) \ .$$

(III-17)

Let us define $\omega = 1 + \frac{t'}{2s'}$ and rearrange, (III-17) then becomes

$$\frac{1}{2}\int_{-1}^{+1} \frac{d\omega P_o(\omega)}{s''-s}\left[\frac{1}{t'+2s(1-\omega)} - \frac{1}{t'+2s''(1-\omega)}\right]$$

$$= \frac{1}{s''-s}\ [\ \frac{1}{2s}\ Q_o(1+\frac{t'}{2s}) - \frac{1}{2s''}\ Q_o(1+\frac{t'}{2s''})] \ .$$

Finally, we find

$$I = \int_o^\infty \frac{ds''}{\pi} \quad \int_{4\beta^2}^\infty \frac{dt'}{\pi}\,\frac{\rho(s'',t')}{s''-s}$$

$$\cdot\ [\frac{1}{2s}\ Q_o(1+\frac{t'}{2s}) - \frac{1}{2s''}\ Q_o(1+\frac{t'}{2s''})] \ .$$

III.2 Exact Solution

Equation (III-5) or (III-16) can be solved exactly provided that $\mathrm{Im}A(s)$ is known for $s \leqslant -\beta^2$.

Assuming

$$\mathrm{Im}\ A(s) = -\pi \sum_{n=1}^{\infty} \gamma_n\ \delta(s+n^2\beta^2/4) \quad ,$$

we find

$$f_B(s) + \int_{-\infty}^{-\beta^2} \frac{ds'}{\pi} \frac{\mathrm{Im}[A(s')-f_B(s')]}{s'-s}$$

$$= \sum_{n=1}^{\infty} \frac{\gamma_n}{s+n^2\beta^2/4} \tag{III-18}$$

with $\gamma_1 = R_1$, since $\mathrm{Im}\ A(s) = \mathrm{Im}\ f_B(s)$ for $-\beta^2 < s \leqslant -\beta^2/4$ from (III-4). By using the unitarity condition (II-8) and (III-18), we can rewrite (III-5) as

$$A(s) = \Sigma \frac{\gamma_n}{s + n^2\beta^2/4} + \int_0^{\infty} \frac{ds'}{\pi} \frac{\sqrt{s'}\ |A(s')|^2}{s'-s-i\epsilon} \quad . \tag{III-19}$$

Let us consider the following Cauchy integral around the contour of Fig. 1,

$$\frac{N_n\ [\gamma_n/(z+n^2\beta^2/4)]}{A(z)} = \frac{1}{2\pi i} \oint \frac{N_n\ [\gamma_n/(z'+n^2\beta^2/4)]}{A(z')(z'-z)} dz' \quad , \tag{III-20}$$

where N_n is a normalization constant independent of z, and

$$\left.\frac{\gamma_n/(z + n^2\beta^2/4)}{A(z)}\right|_{z = -n^2\beta^2/4} = 1 \qquad \text{(III-21)}$$

$$\lim_{z\to\infty} \frac{N_n\,[\gamma_n/(z + n^2\beta^2/4)]}{A(z)} = \text{a constant.} \qquad \text{(III-22)}$$

Then we find that

$$\frac{N_n\,[\gamma_n/(z + n^2\beta^2/4)]}{A(z)} = \frac{1}{2\pi i}\left[\int_0^\infty \frac{N_n\,\gamma_n\,ds'}{(s'-z)(s'+n^2\beta^2/4)A(s'+i\epsilon)}\right.$$

$$\left. + \int_\infty^0 \frac{N_n\,\gamma_n\,ds'}{(s'-z)(s'+n^2\beta^2/4)A(s'-i\epsilon)}\right] + \text{constant}$$

$$= -\int_0^\infty \frac{ds'}{\pi}\,\frac{N_n\,\gamma_n\,\sqrt{s'}}{(s'-z)(s'+n^2\beta^2/4)} + \text{constant},$$

(III-23)

since $\frac{1}{2i}\left[\frac{1}{A(s'+i\epsilon)} - \frac{1}{A(s'-i\epsilon)}\right] = \frac{-\text{Im}\,A(s')}{|\,A(s')\,|^2} = -\sqrt{s'}$,

for $s' > o$, from unitarity condition (II-8). By using (III-21), we obtain

$$\left. \frac{N_n[\gamma_n/(z+n^2\beta^2/4)]}{A(z)} \right|_{z = -n^2\beta^2/4} = N_n$$

$$= - \int_o^\infty \frac{ds'}{\pi} \frac{N_n \gamma_n \sqrt{s'}}{(s'+n^2\beta^2/4)^2} + \text{constant} .$$

(III-24)

By subtracting (III-24) from (III-23), we find that

$$\frac{N_n [\gamma_n/(z+n^2\beta^2/4)]}{A(z)} =$$

$$N_n - N_n\gamma_n(z + n^2\beta^2/4) \int_o^\infty \frac{ds'}{\pi} \frac{\sqrt{s'}}{(s'-z)(s'+n^2\beta^2/4)^2} .$$

Then summing over n from one to infinity, we obtain

$$\frac{\Sigma N_n[\gamma_n/(z + n^2\beta^2/4)]}{A(z)} = \Sigma \frac{N_n[\gamma_n/(z + n^2\beta^2/4)]}{A(z)}$$

$$= \Sigma N_n - \Sigma N_n \gamma_n (z + n^2\beta^2/4)$$

$$\cdot \int_0^\infty \frac{ds'}{\pi} \frac{\sqrt{s'}}{(s'-z)(s' + n^2\beta^2/4)^2} .$$

Then

$$A(z) = \frac{\Sigma N_n [\gamma_n/(z + n^2\beta^2/4)]}{\Sigma N_n - \Sigma N_n\gamma_n(z + n^2\beta^2/4) \int_0^\infty \frac{ds'}{\pi} \frac{\sqrt{s'}}{(s'-z)(s' + n^2\beta^2/4)^2}} .$$

(III-25)

The denominator of (III-25) is normalized to 1 as $z \to \infty$. After evaluating the integral and letting $z \to \infty$, we obtain

$$\Sigma N_n = 1 - \Sigma N_n \gamma_n/n\beta \quad . \qquad \text{(III-26)}$$

Substituting (III-26) into (III-25) and using (III-21), we find a difference equation for the normalization constant N_m,

$$N_m = 1 - \sum_{n=1}^{\infty} N_n \gamma_n/[(n+m) \beta/2] .$$

(III-27)

We can now write the solution (III-25) with (III-26) as

$$A(s) = \frac{\sum_{n=1}^{\infty} N_n\gamma_n/(s + n^2\beta^2/4)}{1 - \sum_{n=1}^{\infty} N_n\gamma_n/(n\beta/2 - i\sqrt{s})}, \tag{III-28}$$

where N_n is determined by (III-27). This solution (III-28) with (III-27) is axactly the same as we shall get later using the N/D method.

III.3 Approximate Solution for Low-Energy Scattering

In general, the left-hand singularities are not known completely. An approximate solution for (III-5) intended to work for low energies has been obtained by Chan and Razavy[20]. This solution involves making the approximation $\text{Im } A(s') \approx \text{Im } f_B(s')$ for $s' < o$, (III-5) then becomes

$$f(s) = f_B(s) + \int_o^\infty \frac{ds'}{\pi} \frac{\sqrt{s'}\,|f(s')|^2}{s'-s-i\epsilon} . \tag{III-29}$$

The symbols $f(s)$ and $\delta(s)$ will be used for the approximate scattering amplitude and phase shift

respectively. We have kept (II-4) and the unitarity condition (II-8) for f(s) and $\delta(s)$. This approximation may be justified for low-energy scattering where the largest contribution comes from the longest-range forces[21] which give rise to the nearest singularities on the negative s axis. We shall merely write down the solution to (III-29)[20],

$$f(s) = \frac{f_B(s)}{1 - \sum_{n=1}^{\infty} \frac{1}{f_{Bn}(s)} \int_0^{\infty} \frac{ds'}{\pi} \frac{\sqrt{s'}\,|f_{Bn}(s')|^2}{s'-s-i\epsilon}} , \qquad \text{(III-30)}$$

where $f_{Bn}(s) = R_n/(s-s_n)$ as defined in (III-2). The low-energy scattering can be expressed in terms of the effective-range parameters:

$$\text{Real}\left[\frac{1}{f(s)}\right] = \sqrt{s}\cot\delta(s) = -a^{-1} + \frac{r_o}{2}s , \qquad \text{(III-31)}$$

where a and r_o are the scattering length and the effective range respectively. For the sum of exponential

potentials (I-1), the effective-range parameters calculated from (III-29), (III-30), and (III-31) are:

$$a^{-1} = \left(\frac{\beta}{2}\right) \left[1 + \frac{1}{2} \sum_m \frac{a_m}{m^2}\right] \Big/ \sum_m \frac{a_m}{m^3} \qquad \text{(III-32)}$$

and

$$r_o = \left(\frac{4}{\beta}\right) \left\{\sum_m a_m \left[\frac{-1}{m^5} + \sum_n a_n\left(\frac{1}{m^3 n^4} - \frac{1}{2m^5 n^2}\right)\right]\right\} \Big/ \sum_m \sum_n \frac{a_m a_n}{m^3 n^3} \; . \qquad \text{(III-33)}$$

Chapter IV. THE N/D METHOD

The dispersion relation for the S-wave amplitude can also be written in the form of N/D, which involves the numerator function and the denominator function. The exact solution which is identical to that for the nonlinear integral equation in Chapter III will be obtained. Again, for low-energy scattering, the approximate solution will be treated by the effective-range parameters.

IV.1 Formulation

The scattering amplitude for the S-wave can be written as the ratio of two functions, N(s) and D(s) (equation (II-4)),

$$A(s) = \frac{g(\sqrt{s},o) - g(-\sqrt{s},o)}{2i\sqrt{s}} \Big/ g(-\sqrt{s},o) = N(s)/D(s), \tag{IV-1}$$

where N(s) is an even function of $\sqrt{s}$. N(s) can have singularities, coming from $g(\sqrt{s},o)$, only for negative s on the first Riemann sheet; whereas the singularities in $g(-\sqrt{s},o)$ are on the second sheet which is irrelevant

to our discussion. $D(s)$ will have a cut on the positive s axis from zero to infinity since it is a function of $\sqrt{s}$; and it has no singularities for negative s because $g(-\sqrt{s},o)$ is analytic on the first Riemann sheet. The bound states, which we exclude, correspond to the zeros of $g(-\sqrt{s},o)$[19]. Hence, the functions $N(z)$ and $D(z)$ are analytic in the z plane except on the negative and the positive real axis respectively. As $z \to \infty$, $N(z) \to o$ and $D(z) \to 1$. From Cauchy's theorem one finds[9]

$$D(s) = 1 + \int_o^\infty \frac{ds'}{\pi} \frac{\operatorname{Im} D(s')}{s'-s-i\epsilon} \tag{IV-2}$$

$$N(s) = \int_{-\infty}^{-\beta^2/4} \frac{ds'}{\pi} \frac{\operatorname{Im} N(s')}{s'-s} \ . \tag{IV-3}$$

Since $A(s) = N(s)/D(s)$, one has

$$\operatorname{Im} D(s) = N(s) \operatorname{Im} [1/A(s)] \tag{IV-4}$$

for positive s , because $N(s)$ is real in this region. For negative s , however, one obtains

$$\operatorname{Im} N(s) = D(s) \operatorname{Im} A(s) . \tag{IV-5}$$

The unitarity condition (II-8) with (IV-4) gives a relation between N and D,

$$\mathrm{Im}\, D(s) = -\sqrt{s}\, N(s) \quad , \text{ for positive } s. \tag{IV-6}$$

Hence, (IV-2) and (IV-3) become coupled integral equations,

$$N(s) = \int_{-\infty}^{-\beta^2/4} \frac{ds'}{\pi} \frac{D(s')\ \mathrm{Im}\, A(s')}{s' - s} \tag{IV-7}$$

and

$$D(s) = 1 - \int_0^{\infty} \frac{ds'}{\pi} \frac{\sqrt{s'}\, N(s')}{s'-s-i\epsilon} \quad . \tag{IV-8}$$

IV.2 Exact Solution

Equations (IV-7) and (IV-8) can be solved exactly if we know Im A(s) for negative s, as in the case of solving the nonlinear integral equation (III-5). Suppose that

$$\text{Im } A(s) = -\pi \sum_{n=1}^{\infty} \gamma_n \, \delta(s + n^2\beta^2/4) \tag{IV-9}$$

for negative s , then we find that

$$N(s) = \sum_n D(-n^2\beta^2/4) \, \gamma_n/(s + n^2\beta^2/4) \,, \tag{IV-10}$$

$$D(s) = 1 - \sum_n D(-n^2\beta^2/4) \, \gamma_n/(n\beta/2 - i\sqrt{s}), \tag{IV-11}$$

and

$$D(-m^2\beta^2/4) = 1 - \sum_n D(-n^2\beta^2/4) \, \gamma_n/[(n+m)\beta/2] \,. \tag{IV-12}$$

Therefore, the solution of (IV-1) is

$$A(s) = \frac{\sum\limits_{n=1}^{\infty} D(-n^2\beta^2/4) \, \gamma_n/(s + n^2\beta^2/4)}{1 - \sum\limits_{n=1}^{\infty} D(-n^2\beta^2/4) \, \gamma_n/(n\beta/2 - i\sqrt{s})} \,. \tag{IV-13}$$

If we identify $D(-n^2\beta^2/4)$ with N_n, this solution is exactly the same as we have obtained for the nonlinear integral equation (Equations (III-28) and (III-27)).

IV.3 Approximate Solution for Low-Energy Scattering

Again, since $\operatorname{Im} A(s)$ is not known completely for negative s, we use the approximation:

$$\operatorname{Im} N(s) \approx \operatorname{Im} f_B(s) = -\pi \sum R_n \, \delta(s-s_n) \; , \quad s < o . \tag{IV-14}$$

Substituting (IV-14) into (IV-3) we find that $N(s) = f_B(s)$. From (IV-6) and (IV-2) we find that the scattering amplitude and the effective-range parameters are given by[20]:

$$f(s) = \frac{f_B(s)}{1 - \int_o^\infty \frac{ds'}{\pi} \frac{\sqrt{s'}\, f_B(s')}{s'-s-i\epsilon}} \tag{IV-15}$$

and

$$\sqrt{s} \quad \cot\,\delta(s) \quad = \quad \text{Real}\ [\frac{1}{f(s)}] \quad = \quad \frac{1}{f_B(s)} - \frac{1}{f_B(s)}$$

$$\cdot \quad P \int_0^\infty \frac{ds'}{\pi} \quad \frac{\sqrt{s}'\ f_B(s')}{s'-s}$$

$$= \ - \ a^{-1} + \frac{r_o}{2}\ s \quad , \qquad \text{(IV-16)}$$

where

$$a^{-1} \quad = \quad (\frac{\beta}{2})[1 + \sum_m \frac{a_m}{m^2}\ ] \Big/ \sum_m \ \frac{a_m}{m^3} \quad , \qquad \text{(IV-17)}$$

and

$$r_o \quad = \quad (\frac{4}{\beta}) \ \frac{\sum_m a_m[\frac{-1}{m^5} + \sum_n a_n(\frac{1}{m^3n^4} - \frac{1}{m^5n^2})]}{\sum_m \sum_n \ \frac{a_m a_n}{m^3n^3}} \quad .$$

$$\text{(IV-18)}$$

Chapter V. APPLICATION TO A CLASS OF SOLVABLE POTENTIALS

We shall consider some of the potentials belonging to the class (I-1) for which the Schrödinger equation can be solved analytically. From (III-28) or (IV-13) exact solutions will be obtained for the exponential potential and the Eckart potential. Exact solutions for the other two potentials, the Morse potential and the Hulthén potential, will be quoted. The effective-range parameters for low-energy scattering will be calculated by the approximate methods, Section III.3 and Section IV.3. For comparison with the approximate solutions, exact solutions will be expanded in powers of the energy and the strength of the potential, λ. Plots of the effective-range parameters against λ will also be given.

V.1 The Exponential Potential

For $a_1 = -\lambda$ and $a_n = o$, $n \neq 1$, we have $V(r) = -\lambda\beta^2 \exp(-\beta r)$, for which the exact scattering amplitude $A(s)$ is of the form[19],

$$A(s) = \frac{1}{2i\sqrt{s}} \left[\frac{J_{2i\sqrt{s}/\beta}(2\sqrt{\lambda})\ \Gamma(1+2i\sqrt{s}/\beta)}{J_{-2i\sqrt{s}/\beta}(2\sqrt{\lambda})\ \Gamma(1-2i\sqrt{s}/\beta)} \lambda^{-2i\sqrt{s}/\beta} - 1 \right] . \tag{V-1}$$

The singularities of $A(s)$ are poles on the physical sheet ($\mathrm{Im}\sqrt{s}$ positive) at the points where $\sqrt{s} = in\beta/2$, poles from $\Gamma(1+2i\sqrt{s}/\beta)$.

For the exact solution let us use (IV-10), (IV-11), and (IV-12). $\mathrm{Im}\, A(s)$ can be easily found from (V-1),

$$\mathrm{Im}\, A(s) = -\frac{\pi\beta}{2} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!(n-1)!} \delta\left(s + \frac{n^2\beta^2}{4}\right), \quad s < o . \tag{V-2}$$

Thus, for

$$\gamma_n = \frac{\beta}{2} \left[\frac{\lambda^n}{n!(n-1)!} \right] ,$$

(IV-10) and (IV-11) become

$$N(s) = \frac{\beta}{2} \Sigma \frac{\lambda^n}{n!(n-1)!} \frac{D(-n^2\beta^2/4)}{(s+n^2\beta^2/4)} \tag{V-3}$$

and

$$D(s) = 1 - \frac{\beta}{2} \Sigma \frac{\lambda^n}{n!(n-1)!} \frac{D(-n^2\beta^2/4)}{(n\beta/2 - i\sqrt{s})} . \tag{V-4}$$

The equation for $D(s)$ can be solved by successively substituting $D(-n^2\beta^2/4)$ into (V-4) and then collecting terms in powers of λ , hence,

$$D(s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (\beta/2)^n \lambda^n}{n!(\beta/2-i\sqrt{s})(2\beta/2-i\sqrt{s})\ldots(n\beta/2-i\sqrt{s})} \, . \tag{V-5}$$

Finding $D(-n^2\beta^2/4)$ from (V-5) and putting it into (V-4), we have

$$D(s) = 1 - \frac{\beta}{2} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!(n\beta/2-i\sqrt{s})} \sum_{j=o}^{\infty} \frac{(-1)^j \lambda^j}{j!(j+n)!} \, . \tag{V-6}$$

Actually, we can go from (V-5) to (V-6) directly by decomposing (V-5) into partial fractions of $(n\beta/2-i\sqrt{s})$ and rearranging to obtain (V-6) finally. If we substitute $D(-n^2\beta^2/4)$ from (V-5) into (V-3), then

$$N(s) = \frac{\beta}{2} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!(s+n^2\beta^2/4)} \sum_{j=o}^{\infty} \frac{(-1)^j \lambda^j}{j!(j+n)!} \, . \tag{V-7}$$

The solutions (V-6) and (V-7) for $D(s)$ and $N(s)$ are exact as can be readily checked by the Jost function for the exponential potential[19],

$$g(-\sqrt{s},o) = J_{-2i\sqrt{s}/\beta}\,(2\lambda^{1/2})\,\Gamma(1-2i\sqrt{s}/\beta)\,\lambda^{+i\sqrt{s}/\beta}$$

$$= \text{the right-hand side of (V-5) or (V-6).}$$

$$\text{(V-8)}$$

We now consider the approximate solutions (Sections III.3 and IV.3). The Born amplitude for the exponential potential has only one pole at $s = -\beta^2/4$, which corresponds to the nearest singularity of $A(s)$, therefore our approximation, in the case of this potential, amounts to retaining the nearest singularity and neglecting all others. From equations (III-32), (III-33), (IV-17), and (IV-18), we calculate the coefficients of the expansion of $\sqrt{s}\cot\delta(s)$ in powers of s, and for comparison the expansion of the exact solution in powers of λ is also included:

nonlinear $$\sqrt{s}\cot\delta(s) = \frac{\beta}{2\lambda}(1 - \frac{\lambda}{2}) + \frac{1}{\lambda\beta}(2 + \lambda)s$$

$$\text{N/D} \qquad \sqrt{s}\,\cot\delta(s) = \frac{\beta}{2\lambda}(1-\lambda) + \frac{2}{\lambda\beta}\,s$$

$$\text{exact} \qquad \sqrt{s}\,\cot\delta(s) = \frac{\beta}{2\lambda}\left(1-\frac{5}{8}\lambda\right) + \frac{1}{\lambda\beta}\left(2+\frac{9}{16}\lambda\right)s \quad . \tag{V-9}$$

V.2 The Morse Potential

There are only two terms in this potential, $a_1 = -2\lambda$ and $a_2 = \lambda$, all other a's are zero, i.e.

$$V(r) = -2\lambda\beta^2\exp(-\beta r) + \lambda\beta^2\exp(-2\beta r) \ .$$

The scattering amplitude is given by[22]

$$A(s) = \frac{1}{2i\sqrt{s}}\left[\frac{F(1/2-\sqrt{\lambda}+i\sqrt{s}/\beta \mid 1+2i\sqrt{s}/\beta \mid 2\sqrt{\lambda})}{F(1/2-\sqrt{\lambda}-i\sqrt{s}/\beta \mid 1-2i\sqrt{s}/\beta \mid 2\sqrt{\lambda})} - 1\right], \tag{V-10}$$

where the F's are the confluent hypergeometric series,

$$F(a \mid b \mid x) = 1 + \frac{a}{b}\,x + \frac{a(a+1)}{b(b+1)}\,\frac{x^2}{2!} + \ldots \tag{V-11}$$

From the analyticity of the confluent hypergeometric series, $A(s)$ has poles at $n + 2i\sqrt{s}/\beta = 0$ $(n = 1, 2, \ldots)$ on the first Riemann sheet (the physical sheet). Approximate solutions for the effective-range parameters and the exact result in powers of λ are obtained as follows:

$$\text{nonlinear} \qquad \sqrt{s} \cot \delta(s) = \frac{4\beta}{15\lambda}(1 - 0.88\lambda) + \frac{28}{25\lambda\beta}(1 + 0.97\lambda)s$$

$$\text{N/D} \qquad \sqrt{s} \cot \delta(s) = \frac{4\beta}{15\lambda}(1 - 1.75\lambda) + \frac{28}{25\lambda\beta}(1 + 0.10\lambda)s$$

$$\text{exact} \qquad \sqrt{s} \cot \delta(s) = \frac{4\beta}{15\lambda}(1 - 1.17\lambda) + \frac{28}{25\lambda\beta}(1 + 0.50\lambda)s$$

(V-12)

V.3 The Hulthén Potential

For this potential $a_n = -\lambda$ for all values of n, and $V(r) = -\lambda\beta^2 \frac{\exp(-\beta r)}{1-\exp(-\beta r)}$. This potential does not satisfy the conditions put on (I-1) (Chapter I). But it is well behaved anywhere except at $r = o$ where the wave function, $u(r)$, is still finite $(u(o) = o$, Section II.1). Thus, the singular point of this potential at $r = o$ will not affect the scattering problem. The scattering amplitude can be written as an infinite product[19],

$$A(s) = \frac{1}{2i\sqrt{s}} \left[\prod_{n=1}^{\infty} \frac{(\sqrt{s} + in\beta/2)^2(\sqrt{s} - i(n^2-\lambda)\beta/2n)}{(s + n^2\beta^2/4)(\sqrt{s} + i(n^2-\lambda)\beta/2n)} - 1 \right],$$

(V-13)

which has poles on the first Riemann sheet at the points where $s = -n^2\beta^2/4$. The Born term,

$$f_B(s) = \frac{\lambda\beta}{2} \Sigma \frac{1}{n(s+n^2\beta^2/4)} , \qquad \text{(V-14)}$$

has an infinite number of poles. Unlike the exponential potential, for this potential the position of the poles of $A(s)$ corresponds to the position of the poles of $f_B(s)$. The effective-range parameters calculated by the approximate methods with an expansion of the exact result as a power series in λ are given below:

nonlinear $\quad \sqrt{s} \cot \delta(s) = \frac{\beta}{2\lambda}(0.83 - 0.68\lambda) + \frac{2}{\lambda\beta}(0.72 + 0.31\lambda)s$

N/D $\quad \sqrt{s} \cot \delta(s) = \frac{\beta}{2\lambda}(0.83 - 1.37\lambda) + \frac{2}{\lambda\beta}(0.72 - 0.28\lambda)s$

exact $\quad \sqrt{s} \cot \delta(s) = \frac{\beta}{2\lambda}(0.83 - 0.72\lambda) + \frac{2}{\lambda\beta}(0.72 + 0.16\lambda)s$

(V-15)

V.4 The Eckart Potential

This potential has the form

$$V(r) = -2\lambda\beta^2 \frac{\exp(-\beta r)}{[1+\lambda\exp(-\beta r)]^2} .$$

Assuming that $\lambda < 1/2$ and from the expansion of $V(r)$ we find $a_n = -2(-1)^{n+1} n\lambda^n$. The scattering amplitude[19],

$$A(s) = \left[\left(\frac{\lambda\beta}{1+\lambda}\right)\frac{1}{(s+\beta^2/4)}\right] \Big/ \left[1 - \left(\frac{\lambda\beta}{1+\lambda}\right)\frac{1}{(\beta/2-i\sqrt{s})}\right] , \tag{V-16}$$

has a single pole on the first sheet at

$$s = -\beta^2/4, \ \sqrt{s} = i\beta/2 \quad ;$$

while the Born amplitude,

$$f_B(s) = \beta \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\lambda^n}{s + n^2\beta^2/4} , \tag{V-17}$$

has an infinite number of poles, and the pole nearest to the origin of the complex z plane corresponds to

the singularity of the scattering amplitude.

The exact solution can be easily found from (IV-10), (IV-11), and (IV-12) with

$$\operatorname{Im} A(s) = -\pi\lambda\beta\delta\,(s+\beta^2/4)$$

for negative s, then

$$N(s) = \left(\frac{\lambda\beta}{1+\lambda}\right)\frac{1}{(s+\beta^2/4)}$$

$$D(s) = 1 - \left(\frac{\lambda\beta}{1+\lambda}\right)\frac{1}{(\beta/2 - i\sqrt{s})} \qquad \text{(V-18)}$$

as expected.

By using (III-32), (III-33), (IV-17), and (IV-18) and keeping the leading terms in powers of λ, the following results are obtained:

$$\text{nonlinear} \qquad \sqrt{s}\cot\delta(s) = \frac{\beta}{4\lambda}\left(1 - \frac{3}{4}\lambda\right) + \frac{1}{\lambda\beta}\left(1 + \frac{23}{16}\lambda\right)s$$

$$\text{N/D} \qquad \sqrt{s}\cot\delta(s) = \frac{\beta}{4\lambda}\left(1 - \frac{7}{4}\lambda\right) + \frac{1}{\lambda\beta}\left(1 + \frac{7}{16}\lambda\right)s$$

$$\text{(V-19)}$$

The exact result for $\sqrt{s}$ cot $\eta(s)$ can be found from (IV-16) or (IV-18) without any expansion:

$$\sqrt{s} \cot \eta(s) = \text{Real}\left[\frac{1}{A(s)}\right] = \frac{\beta}{4\lambda}(1-\lambda) + \frac{1}{\lambda\beta}(1+\lambda)s \quad . \tag{V-20}$$

This is the only potential for which the effective-range theory works exactly.

The following graphs are the plots of the effective-range parameters, $-a^{-1}$ and r_o , against the strength of the potential, λ , for the above four potentials.

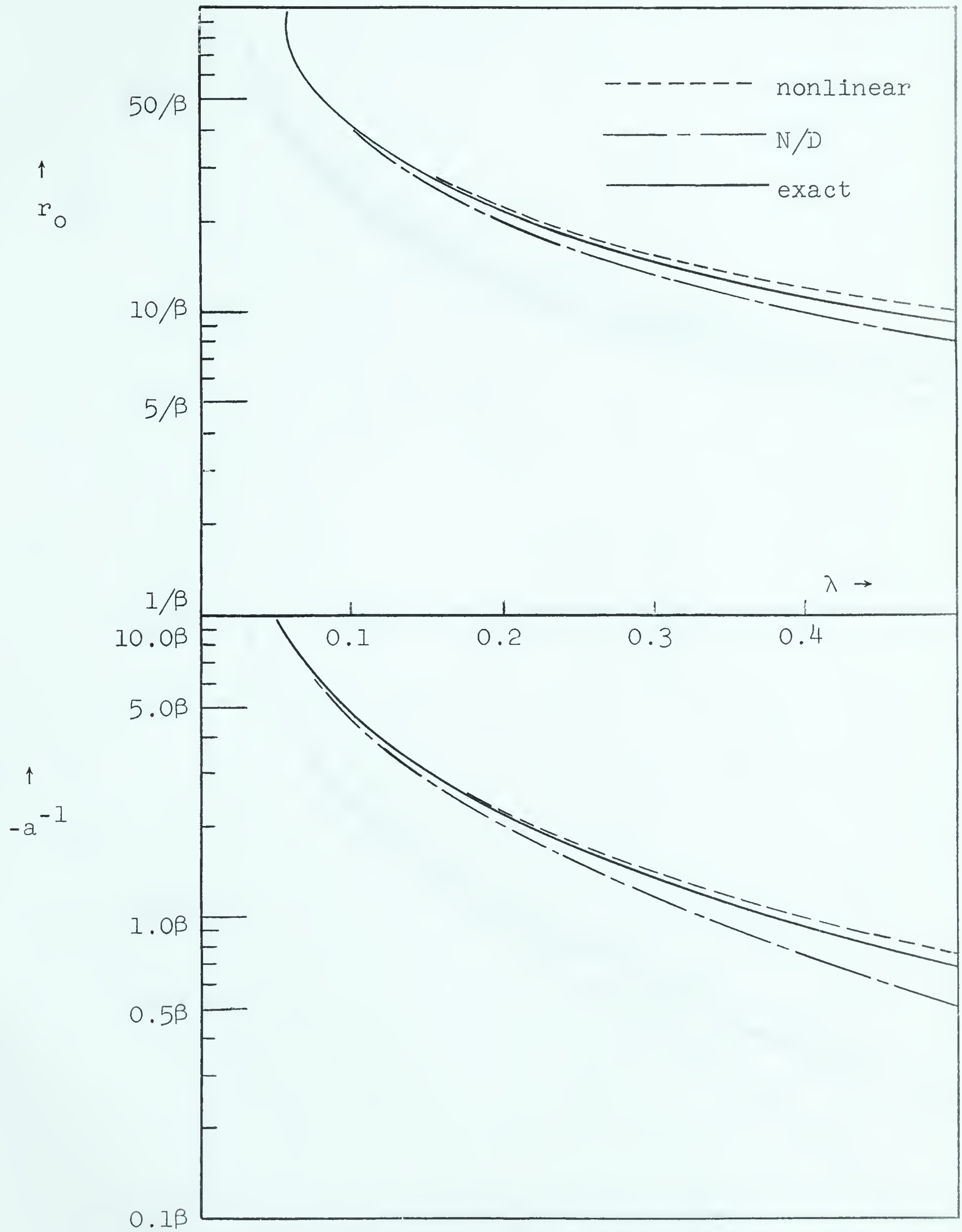

Fig. 2. Effective-Range Parameters vs. λ for the Exponential Potential

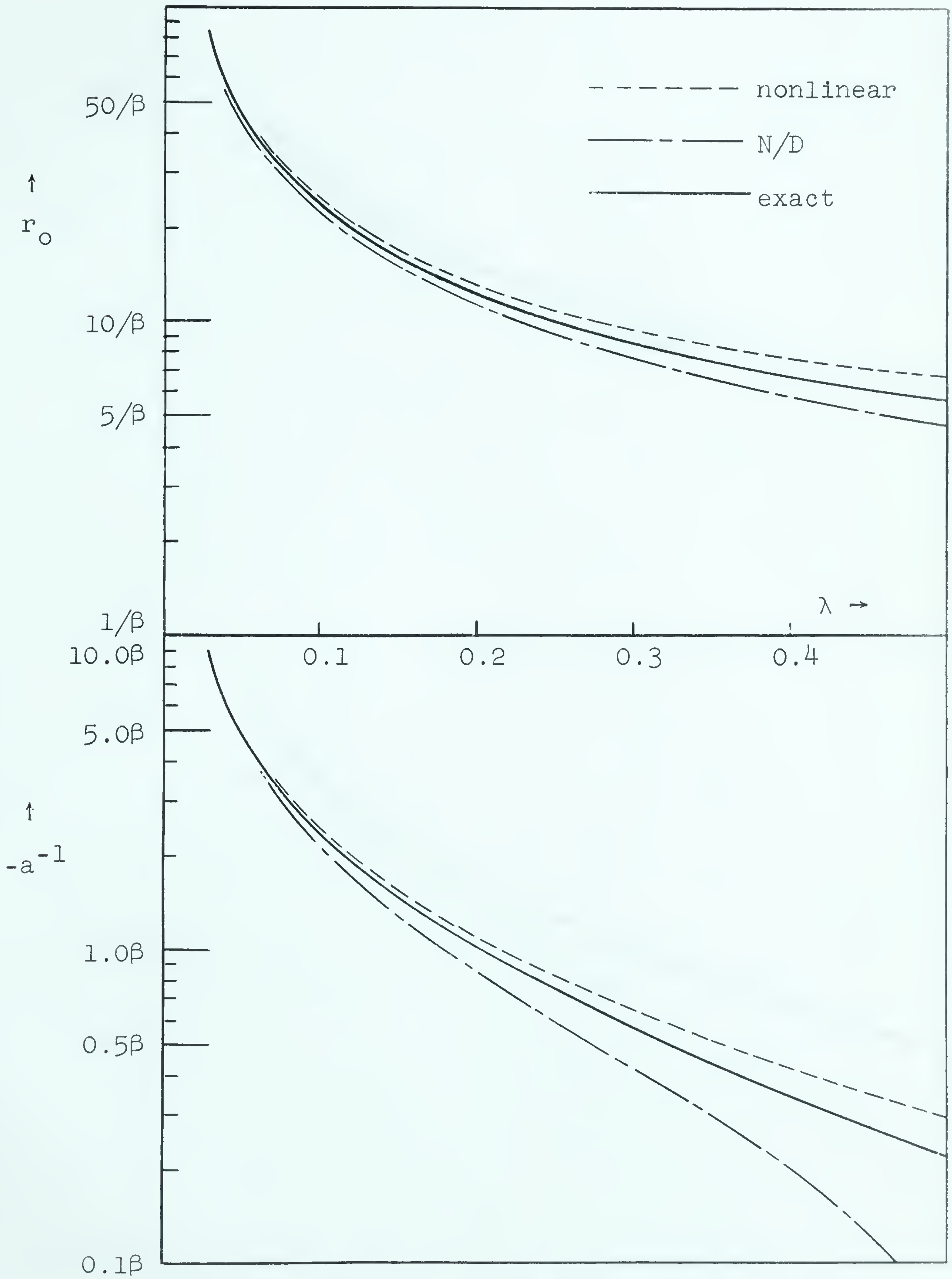

Fig. 3. Effective-Range Parameters vs. λ for the Morse Potential.

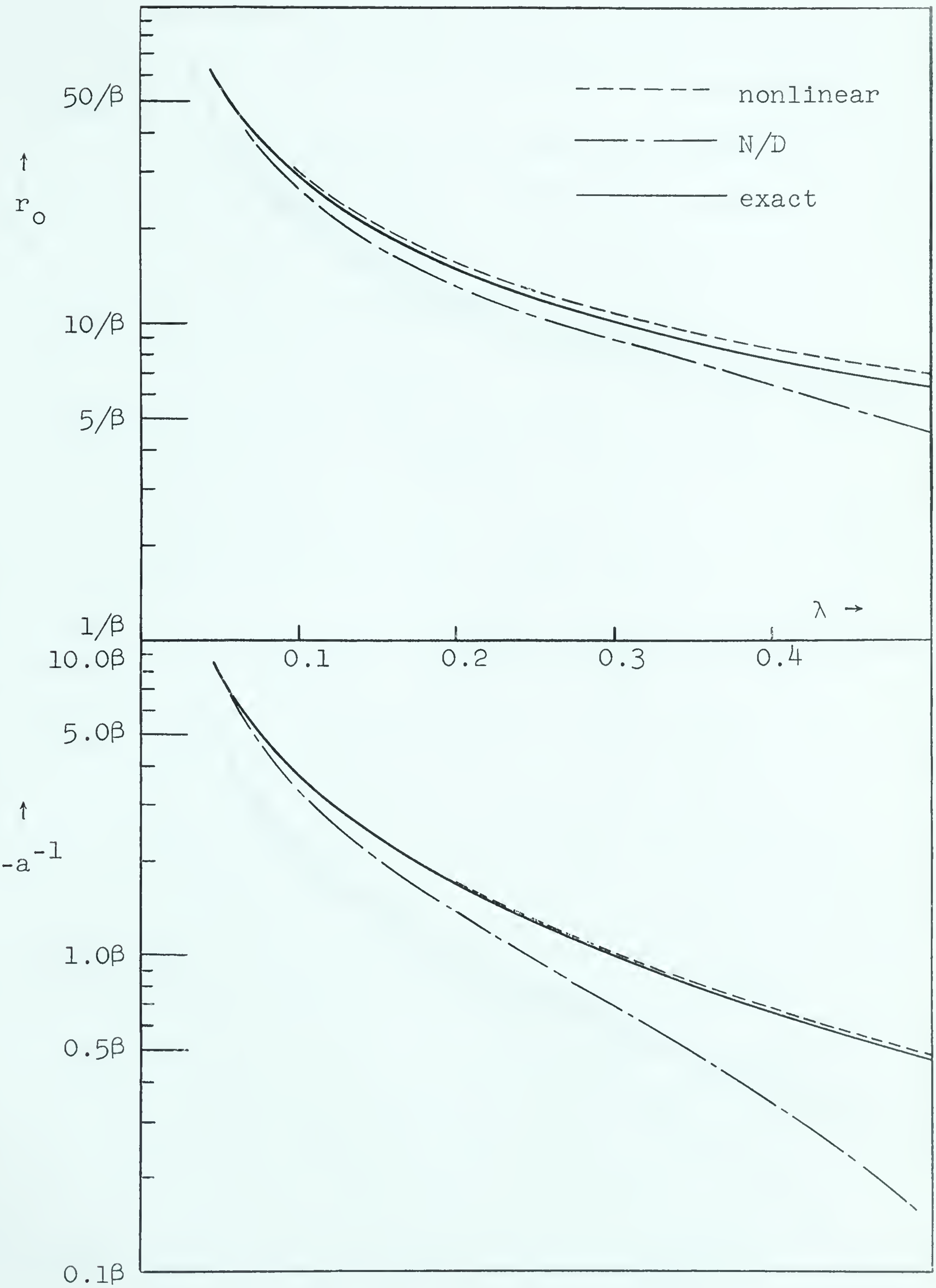

Fig. 4. Effective-Range Parameters vs. λ for the Hulthén Potential

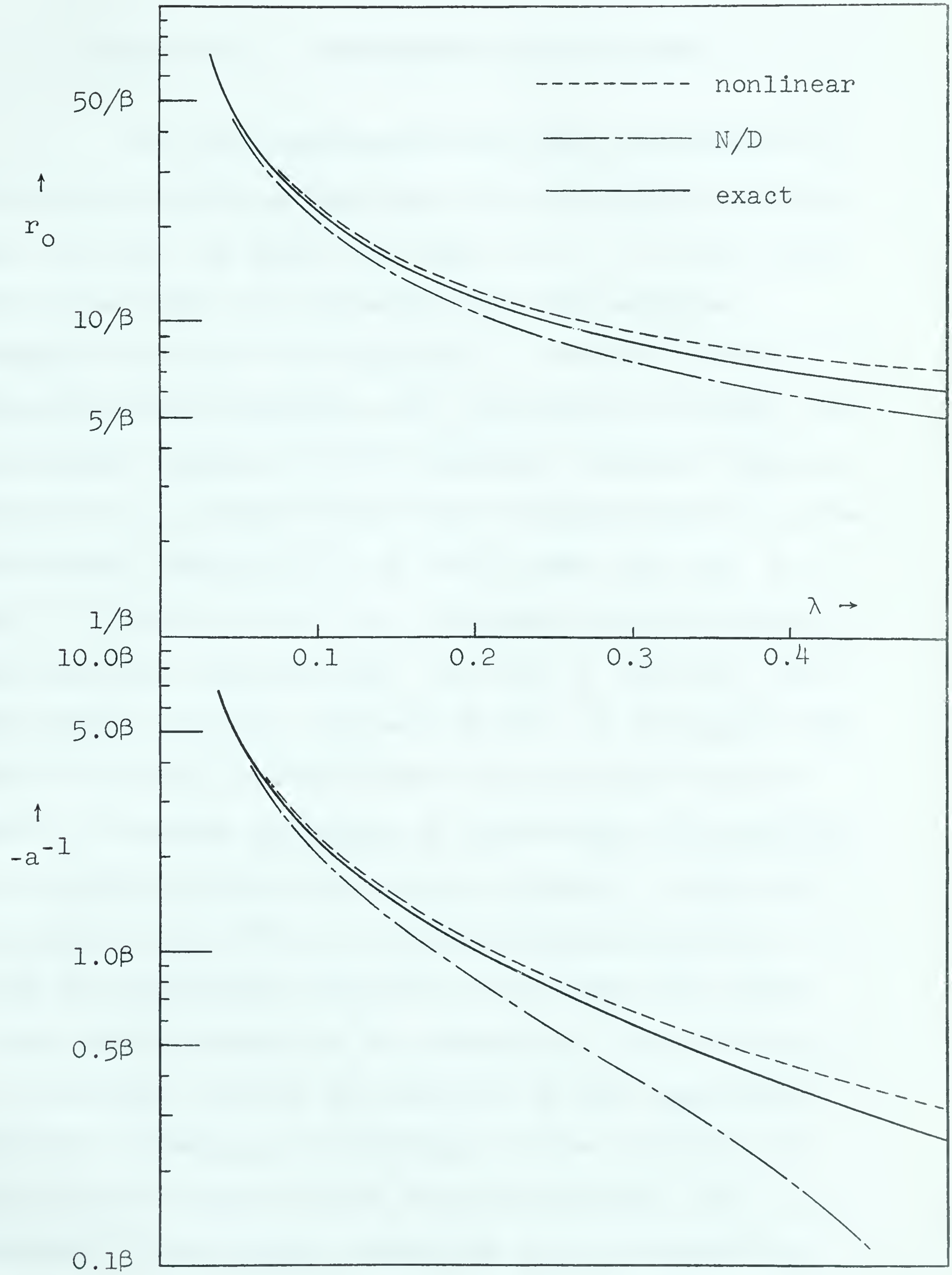

Fig. 5. Effective-Range Parameters vs. λ for the Eckart Potential

Chapter VI. DISCUSSION AND CONCLUSION

We have demonstrated how exact solutions for the S-wave scattering amplitude can be obtained provided that $\text{Im}\, A(s)$ is known for negative s. In this case, the two methods, the N/D method and the nonlinear integral equation, are equivalent. However, for the effective-range parameters for low-energy scattering, the approximate solution for the nonlinear integral equation, Section III.3, works better than the approximation to the N/D method, Section IV.3, as can be seen from Fig. 2 to Fig. 5. This is due to the difference in approximating the left-hand singularities, $\text{Im}\, A(s) \approx \text{Im}\, f_B(s)$ for the integral equation whereas $\text{Im}\, N(s) \approx \text{Im}\, f_B(s)$ for the N/D method, although these two approximations are based on the same assumption of neglecting contributions from distant singularities at low energies. Again from the plots of the effective-range parameters against λ, these two approximate solutions come closer and closer to the exact solution as the strength of the potential, λ, decreases. Hence, the validity of the approximate solutions depends on the strength of the potential and the assumption that distant singularities are not important in low-energy scattering is not necessarily

valid if the strength of the potential is great.

The Mandelstam representation (it becomes the nonlinear integral equation in our case) and unitarity together with the first Born approximation and the left-hand singularities, in effect, completely define the nonrelativistic scattering problem in a way which replaces the Schrödinger equation. The N/D method is exactly equivalent to the nonlinear integral equation, except that the first Born approximation is built into the N equation and is not separated explicitly as in the nonlinear integral equation. However, the Mandelstam representation has a distinct advantage over the Schrödinger equation, that it readily can be generalized to the relativistic case[23].

The ultimate goal of the Mandelstam representation is to solve the scattering problems for different processes without going through the Lagrangian formalism, as in quantum field theory, which contains infinite renormalization constants. Many people have been working in that direction. They make the definite assumption of neglecting distant singularities for low-energy scattering in order to solve their dispersion relations

for given scattering processes. However, since their relativistic scattering problems cannot be solved exactly, the validity of this assumption cannot be checked. In this work, we compare the nonrelativistic scattering results for four potentials, which can be solved analytically by the Schrödinger equation, with the approximate results obtained from the Mandelstam representation under the assumption that for low-energy scattering distant singularities can be neglected. As we mentioned earlier, the validity of this assumption depends on the strength of the potential. Hence, we expect that relativistic scattering will show the same kind of dependence of this assumption upon the interaction strength between the particles.

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APPENDIX

TABLE OF SOLVABLE POTENTIALS

(number of reference referred to Bibliography)

Potentials	References
$V_1 = -\lambda\beta^2 \exp(-\beta r)$	19
$V_2 = -2\lambda\beta^2 \exp(-\beta r)+\lambda\beta^2\exp(-2\beta r)$	22
$V_3 = -\lambda\beta^2 \dfrac{\exp(-\beta r)}{1-\exp(-\beta r)}$	19
$V_4 = -2\lambda\beta^2 \dfrac{\exp(-\beta r)}{[1+\lambda\exp(-\beta r)]^2}$	19
$V_5 = \dfrac{\rho\sigma\{4\rho\sigma+(\rho-\sigma)^2\cosh[(\rho+\sigma)r-2\theta]-(\rho+\sigma)^2\cosh(\rho-\sigma)r\}}{\{\sigma\sinh(\rho r-\theta)-\rho\sinh(\sigma r-\theta)\}^2}$	24
$V_6 = \dfrac{\rho\sigma\{4\rho\sigma+(\rho-\sigma)^2\cosh(\rho+\sigma)r-(\rho+\sigma)^2\cosh[(\rho-\sigma)r+2\theta]\}}{\{\sigma\sinh(\rho r+\theta)-\rho\sinh(\sigma r-\theta)\}^2}$	
$V_7 = \dfrac{-2(\rho/\sigma)(\rho+\sigma)^2\exp[-(\rho+\sigma)r]}{\{1+(\rho/\sigma)\exp[-(\rho+\sigma)r]\}^2}$ (containing V_4)	
$V_8 = \dfrac{-\rho\sigma\{4\rho\sigma+(\rho-\sigma)^2\cosh(\rho+\sigma)r+(\rho+\sigma)^2\cosh[(\rho-\sigma)r-2\theta]\}}{\{\sigma\cosh(\rho r-\theta)+\rho\cosh(\sigma r+\theta)\}^2}$	

Jost Functions, $g(\sqrt{s},o)$, for the Potentials:

$$g_1 = J_{2i\sqrt{s}/\beta}(2\sqrt{\lambda})\ \Gamma(1+2i\sqrt{s}/\beta)\ \lambda^{-i\sqrt{s}/\beta}$$

$$g_2 = F(1/2 - \sqrt{\lambda} + i\sqrt{s}/\beta \mid 1 + 2i\sqrt{s}/\beta \mid 2\sqrt{\lambda})$$

$$g_3 = \prod_{n=1}^{\infty} \frac{n^2 + 2in\sqrt{s}/\beta - \lambda}{n(n + 2i\sqrt{s}/\beta)}$$

$$g_4 = 1 - \left(\frac{2\lambda}{1+\lambda}\right) \frac{1}{1 + 2i\sqrt{s}/\beta}$$

$$g_5 = \frac{2\sqrt{s} + i(\rho+\sigma)}{2\sqrt{s} - i(\rho-\sigma)}$$

$$g_6 = g_7 = g_8 = \frac{2\sqrt{s} + i(\rho-\sigma)}{2\sqrt{s} - i(\rho+\sigma)}$$